Special Cases in Linear Programming

Data Envelopment Analysis

Objective

- To compare technical efficiency of different Decision Making Units (DMU)
- The comparison is made as a function of using inputs optimally, creating an efficient ideal unit.

Paretto – Koopman Efficiency

- A Decision Making Unit (DMU) is not efficient in producing its outputs (from a given amount of inputs) if it can be shown that some distribution of resources will result in the same amount of this output with less of the same resources and no more of any other resources. Conversely, a firm is efficient if this is not possible

An efficient production function

According to Farrell, the production function:

- $Y_0 = \mathcal{Y}(y_1, y_2, \ldots, y_m) = f(x_1, x_2. \ldots, x_k)$

Is efficient is any other vector Y_i produces the same elements such that

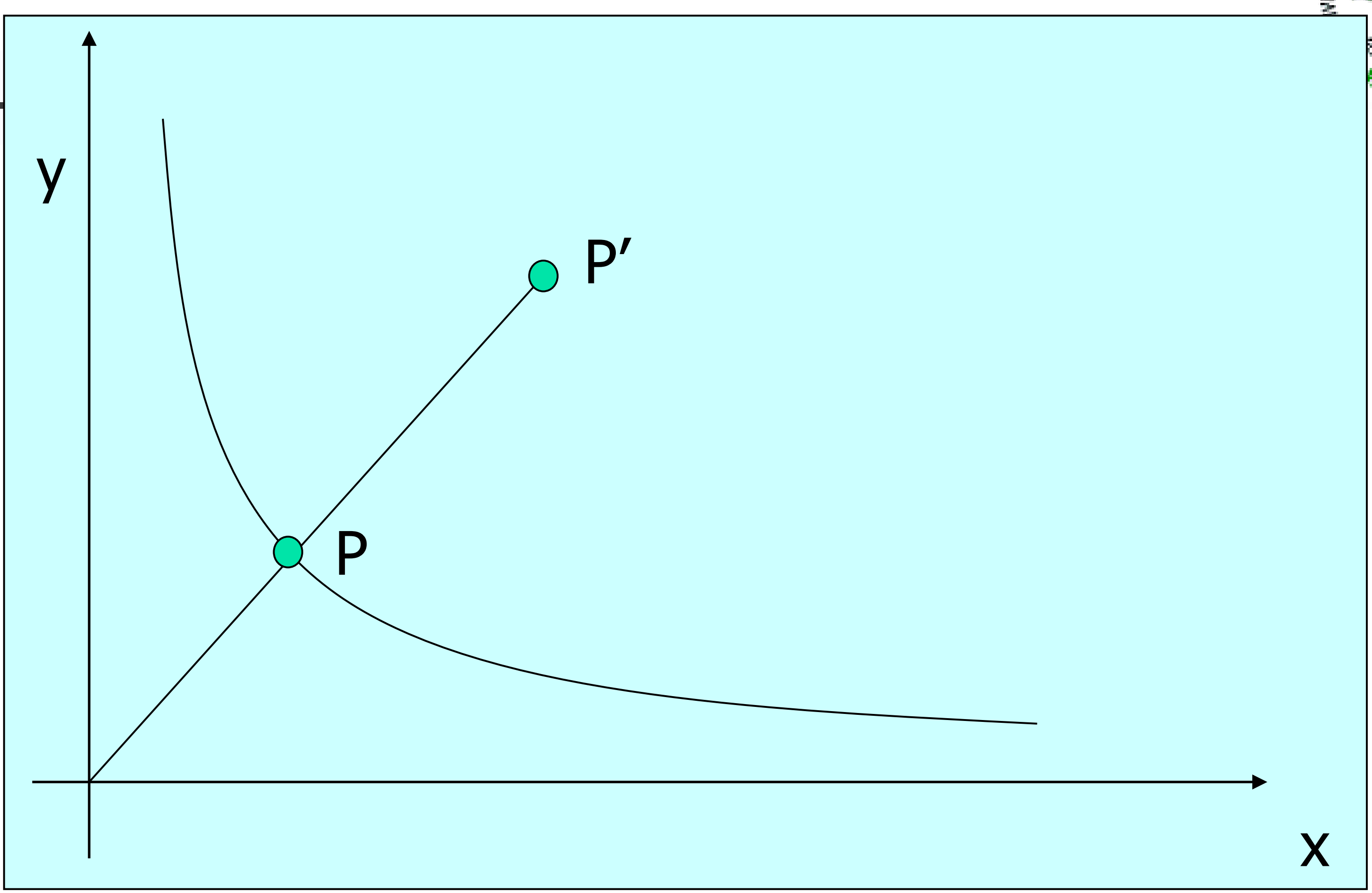
- $Y_i \leq Y_0 \; \forall \; y, x$

Y_i is known as an attainable point of the efficient production function Y_0

If this is true, Y_i is technically possible since it produces at most the same amount of the ideal function Y_0

Characteristics of an efficient function

- **Convexity**: The function has a negative slope that creates an envelope between (0, ∞); (∞, 0)… this condition guarantees that if two points are attainable in practice, then so is any point representing a weighted average of it.
- **Constant returns to scale**: an increase (decrease) in the inputs will produce a proportional increase (decrease) in the outputs.


y
P'
P
x

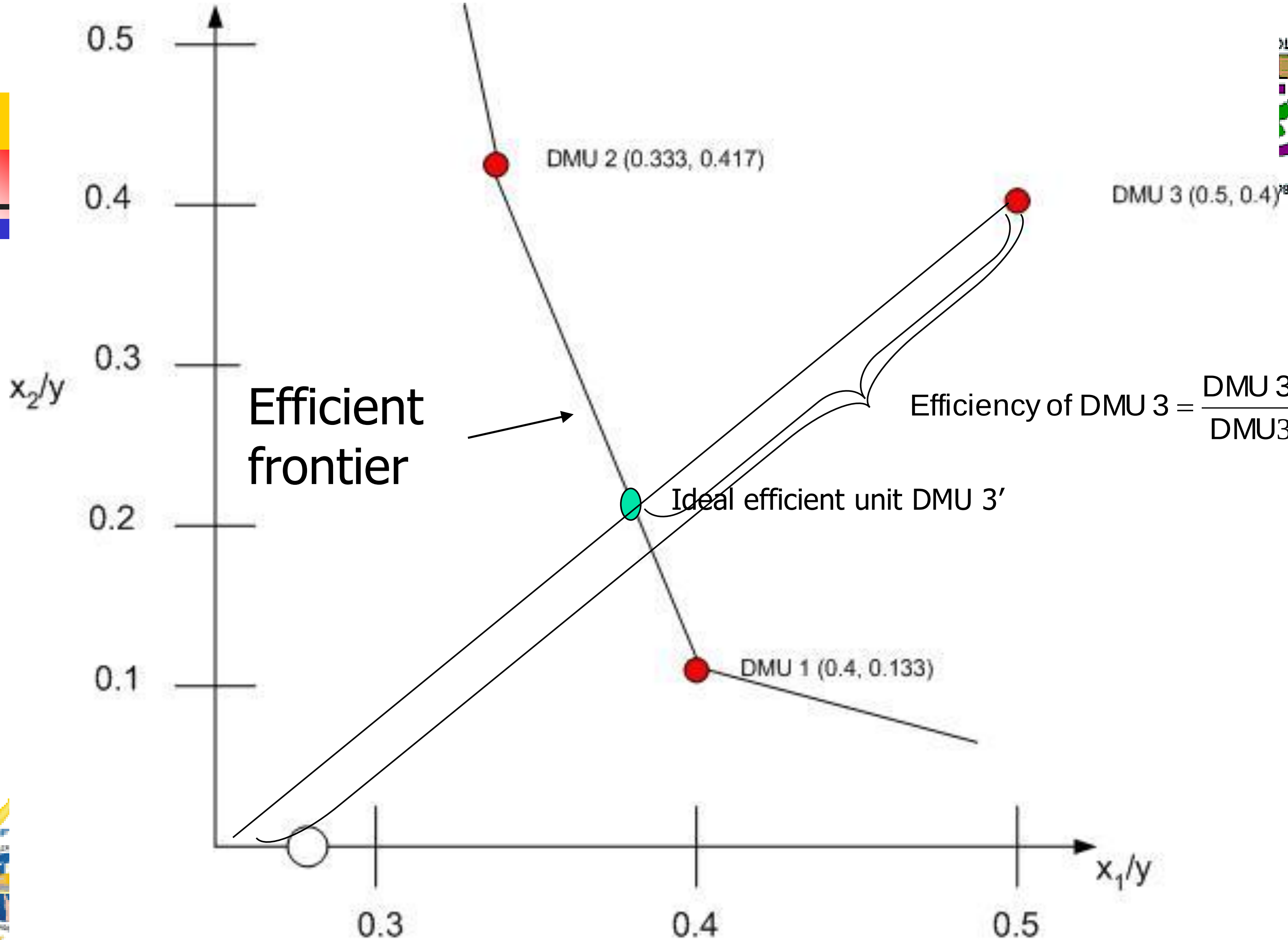
Example

- Three decision making units (DMUs) use two inputs x_1 y x_2 to produce one product y such that:

DMU	y	x_1	x_2
1	15	6	2
2	12	4	5
3	20	10	8

Normalizing the input levels

DMU	x_1/y	x_2/y
1	6/15 = 0.400	2/15 = 0.133
2	4/12 = 0.333	5/12 = 0.417
3	10/20 = 0.500	8/20 = 0.400



0.5
0.4
0.3
0.2
0.1
x_2/y
DMU 2 (0.333, 0.417)
DMU 3 (0.5, 0.4)
Efficient frontier
Efficiency of DMU 3 = $\frac{\text{DMU 3'}}{\text{DMU3}}$
Ideal efficient unit DMU 3’
DMU 1 (0.4, 0.133)
0.3
0.4
0.5
x_1/y

DEA formulation

- Deveolped by Charnes, Cooper andRhodes (1984)
- Non parametric approach based on fractional programming
- Does not require a predefined function

$$\max h_o = \frac{\sum_{r=1}^{s} u_r y_{r,0}}{\sum_{i=1}^{m} v_i x_{i,0}}$$

s.t. :

$$\frac{\sum_{r=1}^{s} u_r y_{r,j}}{\sum_{i=1}^{m} v_i x_{i,j}} \leq 1 \ \forall \ j = 1, 2, ..., n; r = 1, 2, ..., s; i = 1, 2, ..., m$$

$$y_{r,j}, x_{i,j}, u_r, v_i \geq 0$$

Where:

$y_{r,j}$: Is the r-th output of the j-th DMU

$x_{i,j}$: Is the i-th input of the j-th DMU

$u_{r,j}$: Is the weight of the r-th output in the production function of the j-th DMU

$v_{i,j}$: Is the weight of the i-th input in the production function of the j-th DMU

$j=0$: Is the reference unit

In the formulation

- Values of $x_{i,j}$ and $y_{r,j}$ are past observations
- Values of $u_{r,j}$ and $v_{i,j}$ are the decision variables

Formulation as LP

$$\max h_0 = \sum_{r=1}^{s} u_r y_{r,0}$$

s.t. :

$$\sum_{i=1}^{m} v_i x_{i,0} = 1$$

$$\sum_{r=1}^{s} u_r y_{r,j} - \sum_{i=1}^{m} v_i x_{i,j} \leq 0 \ \forall \ j = 1, 2, ..., n$$

$$u_r, v_i \geq 0 \ \forall \ r, j$$

With a dual such that:

$$\min w_0 = q_0$$

s.t.:

$$\sum_{j=1}^{n} p_{0,j} y_{r,j} \geq y_{r,0}$$

$$q_0 x_{i,0} - \sum_{j=1}^{n} p_{0,j} x_{r,j} \geq 0$$

$$p_{0,j} \geq 0; q_0 \text{ not restricted in sign}$$

Input orientation

- A DMU is not efficient if it is possible to maintain the outputs of the unit at a constant level, or increase them, while decreasing any input without augmenting any other input.
- Considering the dual formulation, $p_{0,j}$ will be positive if the corresponding primal constraint defines the DMU as efficient.
- The set of DMUs that contains all the positive $p_{0,j}$ is the reference set for DMU_0

Determining the new efficient unit

$$x_{E,i} = \sum_{j=1}^{n} p_{0,j} x_{i,j}; \quad \text{for } i = 1, 2, \ldots, m$$

$$y_{E,r} = \sum_{j=1}^{n} p_{0,j} y_{r,j}; \quad \text{for } r = 1, 2, \ldots, s$$

Example

- For the previous case:

DMU	y	x_1	X_2
1	15	6	2
2	12	4	5
3	20	10	8

Formulation and solution with QSB

DMU 1

DMU	y	x_1	X_2
1	15	6	2
2	12	4	5
3	20	10	8

$$\max h_0 = \sum_{r=1}^{s} u_r y_{r,0}$$

s.t.:

$$\sum_{i=1}^{m} v_i x_{i,0} = 1$$

$$\sum_{r=1}^{s} u_r y_{r,j} - \sum_{i=1}^{m} v_i x_{i,j} \leq 0 \;\forall\; j = 1, 2, \ldots, n$$

$$u_r, v_i \geq 0 \;\forall\; r, j$$

Max. $15u_1$

s.t.:

$$6v_1 + 2v_3 = 1$$

$$15u_1 - 6v_1 - 2v_3 \leq 0$$

$$12u_1 - 4v_1 - 5v_3 \leq 0$$

$$20u_1 - 10v_1 - 8v_3 \leq 0$$

DMU	y	x_1	X_2
1	15	6	2
2	12	4	5
3	20	10	8

DMU 2

$$\max h_0 = \sum_{r=1}^{s} u_r y_{r,0}$$

s.t :

$$\sum_{i=1}^{m} v_i x_{i,0} = 1$$

$$\sum_{r=1}^{s} u_r y_{r,j} - \sum_{i=1}^{m} v_i x_{i,j} \leq 0 \ \forall \ j = 1, 2, \ldots, n$$

$$u_r, v_i \geq 0 \ \forall \ r, j$$

Max. $12u_1$

s.t.:

$$4v_1 + 5v_3 = 1$$

$$15u_1 - 6v_1 - 2v_3 \leq 0$$

$$12u_1 - 4v_1 - 5v_3 \leq 0$$

$$20u_1 - 10v_1 - 8v_3 \leq 0$$

DMU 3

DMU	y	x_1	X_2
1	15	6	2
2	12	4	5
3	20	10	8

$$\max h_0 = \sum_{r=1}^{s} u_r y_{r,0}$$

s.t.:

$$\sum_{i=1}^{m} v_i x_{i,0} = 1$$

$$\sum_{r=1}^{s} u_r y_{r,j} - \sum_{i=1}^{m} v_i x_{i,j} \leq 0 \; \forall \; j = 1, 2, \ldots, n$$

$$u_r, v_i \geq 0 \; \forall \; r, j$$

Max. $20u_1$

s.t.:

$$10v_1 + 8v_3 = 1$$

$$15u_1 - 6v_1 - 2v_3 \leq 0$$

$$12u_1 - 4v_1 - 5v_3 \leq 0$$

$$20u_1 - 10v_1 - 8v_3 \leq 0$$

DMU1

Variable -->	u1	v1	v2	Direction	R. H. S.
Maximize	15				
Referencia 1		6	2	=	1
DMU1	15	-6	-2	<=	0
DMU2	12	-4	-5	<=	0
DMU3	20	-10	-8	<=	0
LowerBound	0	0	0		
UpperBound	M	M	M		
VariableType	Continuous	Continuous	Continuous		

	18:46:41		Friday	March	03	2006		
	Decision Variable	Solution Value	Unit Cost or Profit c(j)	Total Contribution	Reduced Cost	Basis Status	Allowable Min. c(j)	Allowable Max. c(j)
1	u1	0.0667	15.0000	1.0000	0	basic	0	M
2	v1	0.1545	0	0	0	basic	0	13.7500
3	v2	0.0364	0	0	0	basic	-4.5833	0
	Objective	Function	(Max.) =	1.0000	(Note:	Alternate	Solution	Exists!!)
	Constraint	Left Hand Side	Direction	Right Hand Side	Slack or Surplus	Shadow Price	Allowable Min. RHS	Allowable Max. RHS
1	Referencia 1	1.0000	=	1.0000	0	1.0000	0	M
2	DMU1	0.0000	<=	0	0	1.0000	-0.1667	1.5962
3	DMU2	0.0000	<=	0	0	0	-1.7000	0.1333
4	DMU3	-0.5030	<=	0	0.5030	0	-0.5030	M

Relative efficiency

$p_{o,j}$

DMU2

Variable -->	u1	v1	v2	Direction	R. H. S.
Maximize	12				
Referencia 1		4	5	=	1
DMU1	15	-6	-2	<=	0
DMU2	12	-4	-5	<=	0
DMU3	20	-10	-8	<=	0
LowerBound	0	0	0		
UpperBound	M	M	M		
VariableType	Continuous	Continuous	Continuous		

	19:01:45		Friday	March	03	2006		
	Decision Variable	Solution Value	Unit Cost or Profit c(j)	Total Contribution	Reduced Cost	Basis Status	Allowable Min. c(j)	Allowable Max. c(j)
1	u1	0.0833	12.0000	1.0000	0	basic	0	M
2	v1	0.1932	0	0	0	basic	-3.5200	0
3	v2	0.0455	0	0	0	basic	0	4.4000
	Objective	Function	(Max.) =	1.0000	(Note:	Alternate	Solution	Exists!!)
	Constraint	Left Hand Side	Direction	Right Hand Side	Slack or Surplus	Shadow Price	Allowable Min. RHS	Allowable Max. RHS
1	Referencia 1	1.0000	=	1.0000	0	1.0000	0	M
2	DMU1	0.0000	<=	0	0	0	-0.2500	0.7685
3	DMU2	0.0000	<=	0	0	1.0000	-0.6800	0.2000
4	DMU3	-0.6288	<=	0	0.6288	0	-0.6288	M

Relative efficiency

$p_{o,j}$

DMU3

Variable -->	u1	v1	v2	Direction	R. H. S.
Maximize	20				
Referencia 1		10	8	=	1
DMU1	15	-6	-2	<=	0
DMU2	12	-4	-5	<=	0
DMU3	20	-10	-8	<=	0
LowerBound	0	0	0		
UpperBound	M	M	M		
VariableType	Continuous	Continuous	Continuous		

	19:06:09		Friday	March	03	2006		
	Decision Variable	Solution Value	Unit Cost or Profit c(j)	Total Contribution	Reduced Cost	Basis Status	Allowable Min. c(j)	Allowable Max. c(j)
1	u1	0.0363	20.0000	0.7261	0	basic	0	M
2	v1	0.0842	0	0	0	basic	-4.6667	3.7500
3	v2	0.0198	0	0	0	basic	-3.0000	3.7333
	Objective	Function	(Max.) =	0.7261				
	Constraint	Left Hand Side	Direction	Right Hand Side	Slack or Surplus	Shadow Price	Allowable Min. RHS	Allowable Max. RHS
1	Referencia 1	1.0000	=	1.0000	0	0.7261	0	M
2	DMU1	0.0000	<=	0	0	0.5941	-0.1000	0.4611
3	DMU2	0.0000	<=	0	0	0.9241	-0.4250	0.0800
4	DMU3	-0.2739	<=	0	0.2739	0	-0.2739	M

Relative efficiency

$p_{o,j}$

Unit	Efficiency
DMU 1	100.00 %
DMU 2	100.00 %
DMU 3	72.61 %

Efficient unit

$X_{E,1}$ = 0.5941*6 + 0.9241*4 = 7.2610

$X_{E,2}$ = 0.5941*2 + 0.9241*5 = 5.8087

$y_{E,1}$ = 0.5941*15 + 0.9241*12=20.0007

Constraint	Left Hand Side	Direction	Right Hand Side	Slack or Surplus	Shadow Price	A M
Referencia 1	1.0000	=	1.0000	0	0.7261	
DMU1	0.0000	<=	0	0	0.5941	
DMU2	0.0000	<=	0	0	0.9241	
DMU3	-0.2739	<=	0	0.2739	0	

DMU	y	x_1	X_2
1	15	6	2
2	12	4	5
3	20	10	8

Validating the efficient DMU

Variable -->	u1	v1	v2	Direction	R. H. S.
Maximize	20				
Referencia 1		7.261	5.8087	=	1
DMU1	15	-6	-2	<=	0
DMU2	12	-4	-5	<=	0
DMU3	20	-7.261	-5.8087	<=	0
LowerBound	0	0	0		
UpperBound	M	M	M		
VariableType	Continuous	Continuous	Continuous		

	19:13:38		Friday	March	03	2006		
	Decision Variable	Solution Value	Unit Cost or Profit c(j)	Total Contribution	Reduced Cost	Basis Status	Allowable Min. c(j)	Allowable Max. c(j)
1	u1	0.0500	20.0000	1.0000	0	basic	0	M
2	v1	0.1159	0	0	0	basic	-4.6666	3.7502
3	v2	0.0273	0	0	0	basic	-3.0001	3.7332
	Objective	Function	(Max.) =	1.0000				
	Constraint	Left Hand Side	Direction	Right Hand Side	Slack or Surplus	Shadow Price	Allowable Min. RHS	Allowable Max. RHS
1	Referencia 1	1.0000	=	1.0000	0	1.0000	0	M
2	DMU1	0.0000	<=	0	0	0.5941	-0.1377	0.0001
3	DMU2	0.0000	<=	0	0	0.9241	-0.5853	0.0000
4	DMU3	0.0000	<=	0	0.0000	0	0.0000	M

Output orientation

- Developed by Bessent and Bessent (1988)
- This approach considers the difficulties in allocating resources in a competitive market
- A DMU is not efficient if it is possible to augment any output without increasing any input, and without decreasing any other output
- This approach presents a formulation slightly different from the dual

General formulation

$$\max z_0$$

$$s.t.:$$

$$y_{r,0} z_0 - \sum_{j=1}^{n} y_{r,j}\delta_j + S_r^{+} = 0; \text{ for } r = 1, 2, ..., s$$

$$\sum_{j=1}^{n} x_{i,j}\delta_j + S_r^{-} = x_{i,0}; \text{ for } i = 1, 2, ..., m$$

$$x_{i,j}, y_{r,j}, \delta_j, S_r^{+}, S_r^{-} \geq 0$$

Where:

z_0 : The efficiency will be given by $h_0=1/z_0$

δ_j : Is the weight for DMU j. Is the decision variable of the problem.

S_r^+, S_r^- : Are slack variables of the constraints

Defining the efficient unit

$$y_{E,r} = z_0 y_{0,r} + S_r^+$$

$$x_{E,r} = x_{0,i} - S_r^-$$

Example

- For the previous case:

DMU	y	x_1	X_2
1	15	6	2
2	12	4	5
3	20	10	8

General formulation and solution in WinQSB

DMU 1

$$\max z_0$$

s.t. :

$$y_{r,0}z_0 - \sum_{j=1}^{n} y_{r,j}\delta_j + S_r^+ = 0; \text{ for } r = 1, 2, \ldots, s$$

$$\sum_{j=1}^{n} x_{i,j}\delta_j + S_r^- = x_{i,0}; \text{ for } i = 1, 2, \ldots, m$$

$$x_{i,j}, y_{r,j}, \delta_j, S_r^+, S_r^- \geq 0$$

DMU	y	x_1	X_2
1	15	6	2
2	12	4	5
3	20	10	8

Max z

s.t.:

$$15z - 15d_1 - 12d_2 - 20d_3 \leq 0$$

$$6d_1 + 4d_2 + 10d_3 \leq 6$$

$$2d_1 + 5d_2 + 8d_3 \leq 2$$

DMU 2

$$\max z_0$$

s.t. :

$$y_{r,0} z_0 - \sum_{j=1}^{n} y_{r,j}\delta_j + S_r^+ = 0; \text{ for } r = 1, 2, \ldots, s$$

$$\sum_{j=1}^{n} x_{i,j}\delta_j + S_r^- = x_{i,0}; \text{ for } i = 1, 2, \ldots, m$$

$$x_{i,j}, y_{r,j}, \delta_j, S_r^+, S_r^- \geq 0$$

DMU	y	x_1	X_2
1	15	6	2
2	12	4	5
3	20	10	8

Max z

s.t.:

$$12z - 15d_1 - 12d_2 - 20d_3 \leq 0$$

$$6d_1 + 4d_2 + 10d_3 \leq 4$$

$$2d_1 + 5d_2 + 8d_3 \leq 5$$

DMU 3

$$\max z_0$$
s.t.:
$$y_{r,0}z_0 - \sum_{j=1}^{n} y_{r,j}\delta_j + S_r^+ = 0;\ \text{for } r = 1, 2, \ldots, s$$
$$\sum_{j=1}^{n} x_{i,j}\delta_j + S_r^- = x_{i,0};\ \text{for } i = 1, 2, \ldots, m$$
$$x_{i,j}, y_{r,j}, \delta_j, S_r^+, S_r^- \geq 0$$

DMU	y	x_1	X_2
1	15	6	2
2	12	4	5
3	20	10	8

Max z
s.t.:
$$20z - 15d_1 - 12d_2 - 20d_3 \leq 0$$
$$6d_1 + 4d_2 + 10d_3 \leq 10$$
$$2d_1 + 5d_2 + 8d_3 \leq 8$$

DMU 1

Variable -->	z	d1	d2	d3	Direction	R. H. S.
Maximize	1					
y1	15	-15	-12	-20	<=	0
x1		6	4	10	<=	6
x2		2	5	8	<=	2
LowerBound	0	0	0	0		
UpperBound	M	M	M	M		
VariableType	Continuous	Continuous	Continuous	Continuous		

	21:13:30		Friday	March	03	2006		
	Decision Variable	Solution Value	Unit Cost or Profit c(j)	Total Contribution	Reduced Cost	Basis Status	Allowable Min. c(j)	Allowable Max. c(j)
1	z	1.0000	1.0000	1.0000	0	basic	0	M
2	d1	1.0000	0	0	0	basic	-0.6148	0.2000
3	d2	0	0	0	0	basic	-0.1333	1.7000
4	d3	0	0	0	-0.5030	at bound	-M	0.5030
	Objective	Function	(Max.) =	1.0000				
	Constraint	Left Hand Side	Direction	Right Hand Side	Slack or Surplus	Shadow Price	Allowable Min. RHS	Allowable Max. RHS
1	y1	0	<=	0	0	0.0667	-15.0000	M
2	x1	6.0000	<=	6.0000	0	0.1545	1.6000	6.0000
3	x2	2.0000	<=	2.0000	0	0.0364	2.0000	7.5000

DMU 2

Variable -->	z	d1	d2	d3	Direction	R. H. S.
Maximize	1					
y1	12	-15	-12	-20	<=	0
x1		6	4	10	<=	4
x2		2	5	8	<=	5
LowerBound	0	0	0	0		
UpperBound	M	M	M	M		
VariableType	Continuous	Continuous	Continuous	Continuous		

	21:15:58		Friday	March	03	2006		
	Decision Variable	Solution Value	Unit Cost or Profit c(j)	Total Contribution	Reduced Cost	Basis Status	Allowable Min. c(j)	Allowable Max. c(j)
1	z	1.0000	1.0000	1.0000	0	basic	0	M
2	d1	0	0	0	-0.2500	at bound	-M	0.2500
3	d2	1.0000	0	0	0	basic	-0.1667	M
4	d3	0	0	0	-0.8333	at bound	-M	0.8333
	Objective	Function	(Max.) =	1.0000				
	Constraint	Left Hand Side	Direction	Right Hand Side	Slack or Surplus	Shadow Price	Allowable Min. RHS	Allowable Max. RHS
1	y1	0	<=	0	0	0.0833	-12.0000	M
2	x1	4.0000	<=	4.0000	0	0.2500	0	4.0000
3	x2	5.0000	<=	5.0000	0	0	5.0000	M

DMU 3

Variable -->	z	d1	d2	d3	Direction	R. H. S.
Maximize	1					
y1	20	-15	-12	-20	<=	0
x1		6	4	10	<=	10
x2		2	5	8	<=	8
LowerBound	0	0	0	0		
UpperBound	M	M	M	M		
VariableType	Continuous	Continuous	Continuous	Continuous		

	21:21:01		Friday	March	03	2006		
	Decision Variable	Solution Value	Unit Cost or Profit c(j)	Total Contribution	Reduced Cost	Basis Status	Allowable Min. c(j)	Allowable Max. c(j)
1	z	1.3773	1.0000	1.3773	0	basic	0	M
2	d1	0.8182	0	0	0	basic	-0.4611	0.1500
3	d2	1.2727	0	0	0	basic	-0.1000	1.2750
4	d3	0	0	0	-0.3773	at bound	-M	0.3773
	Objective	Function	(Max.) =	1.3773				
	Constraint	Left Hand Side	Direction	Right Hand Side	Slack or Surplus	Shadow Price	Allowable Min. RHS	Allowable Max. RHS
1	y1	0	<=	0	0	0.0500	-27.5455	M
2	x1	10.0000	<=	10.0000	0	0.1159	6.4000	24.0000
3	x2	8.0000	<=	8.0000	0	0.0273	3.3333	12.5000

Efficient unit

$$y_{E,r} = z_0 y_{0,r} + s_r^{+}$$

$$x_{E,r} = x_{0,i} - s_r^{-}$$

Objective	Function	(Max.) =	1.3773		
Constraint	Left Hand Side	Direction	Right Hand Side	Slack or Surplus	Sh P
1 y1	0	<=	0	0	0.
2 x1	10.0000	<=	10.0000	0	0.
3 x2	8.0000	<=	8.0000	0	0.

Efficiency of DMU 3 = 1/1.3773 = 0.7261

$y_{E,1}$= 1.3773*20 + 0 = 27.546

$x_{E,1}$=10 – 0 = 10

$x_{E,2}$ = 8 – 0 = 8

DMU 3 Eficiente

Variable -->	z	d1	d2	d3	Direction	R. H. S.
Maximize	1					
y1	27.546	-15	-12	-27.546	<=	0
x1		6	4	10	<=	10
x2		2	5	8	<=	8
LowerBound	0	0	0	0		
UpperBound	M	M	M	M		
VariableType	Continuous	Continuous	Continuous	Continuous		

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	21:28:49		Friday	March	03	2006		
	Decision Variable	Solution Value	Unit Cost or Profit c(j)	Total Contribution	Reduced Cost	Basis Status	Allowable Min. c(j)	Allowable Max. c(j)
1	z	1.0000	1.0000	1.0000	0	basic	0	M
2	d1	0	0	0	0.0000	at bound	-M	0.0000
3	d2	0	0	0	0	basic	-0.0356	0.0000
4	d3	1.0000	0	0	0	basic	0.0000	0.0891
	Objective	Function	(Max.) =	1.0000				
	Constraint	Left Hand Side	Direction	Right Hand Side	Slack or Surplus	Shadow Price	Allowable Min. RHS	Allowable Max. RHS
1	y1	0	<=	0	0	0.0363	-27.5460	M
2	x1	10.0000	<=	10.0000	0	0.0842	6.4000	10.0000
3	x2	8.0000	<=	8.0000	0	0.0198	8.0000	12.5000

Find the efficienciy of the DMUs shown in the table:

Plant	Resources				Sales (Millions of US$)
	Engines (Units)	Financial Resources (Thousands of US$)	Technology (Thousands of US$)	Other resources (Thousands of US$)	
Germany	400	230	125	200	70
France	900	270	200	230	130
Belgium	200	120	150	230	70
Netherlands	500	200	320	420	80