Mathematical optimization

Optimization models

- Their objective is to select the best decision from a number of possible alternatives, without a complete enumeration of them.
- Optimization theory is a branch of applied mathematics that explains these problems.

Optimization methods: mathematical programming

- Objective:
  - To find the best point that optimizes an economic model
- General formulation:
  - Optimize y(**x**)

    Subject to $f(\mathbf{x}) \geq 0 \ \forall$ i, $\mathbf{x} = (x_1, x_2, \ldots, x_n)$
- Métodos:
  - Analytic methods, Linear Programming, metaheuristics, combinatory methods.

Optimization methods: variational methdos

- Objective:
  - To find the best function that optimzes an economic model
- General formulation
  - Optimize $I[y(x)] = \int F[y(x), y'(x)]dx$

    Subject to algebraic and mathematical constraints
- Methods:
  - Variational calculus, continuous models

General theory of maxima and minima

- Not constrained models
- Its objective is to find the extreme points of a function.
- Teorems:
  - If a fuction is continuous in a closed interval, it has a maxima or minima in the interior or extremes of the interval.
  - A continuous function has a maxima or minima in the interior of a region only if the n derivative is 0 (inflection point) or it doesn´t exist (dicountinous point).

Global and local optima

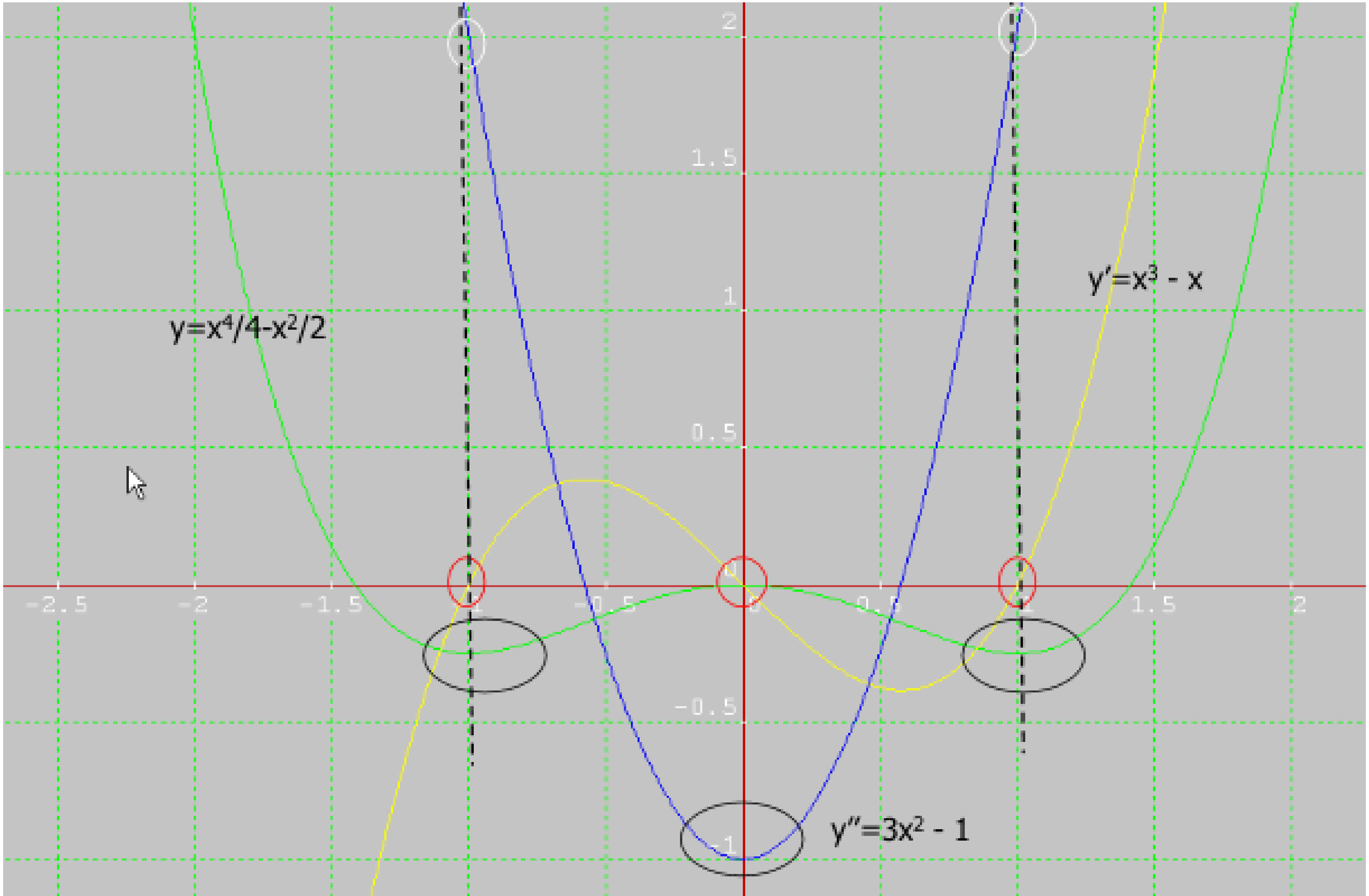
- An optima will be local if it has a maxima or minima in the closed interval [a, b]
- An optima will be global is it has a maxima or minima in the interval

  [-∞, ∞]
- If the local optima is the same as the global optima, then the function has an exact optima)

Sufficient conditions for the optima of an independent variable.

For a continuous function $f(x)$, if:

- $f'(x)\ \exists\ \forall\ x \in \Re$
- X* is critical in $f'(x^*) = 0$
- $f''(x^*) = \begin{cases} > 0 \rightarrow \text{min} \\ < 0 \rightarrow \text{max} \\ = 0 \rightarrow \text{no definition} \end{cases}$
- Si $f''(x^*) = 0$, the n higher order derivatives are examined until $f^n(x^*) \neq 0$
- If n es even: $f^n(x^*) = \begin{cases} > 0 \rightarrow \text{min} \\ < 0 \rightarrow \text{max} \end{cases}$
- If n is odd: saddle point.



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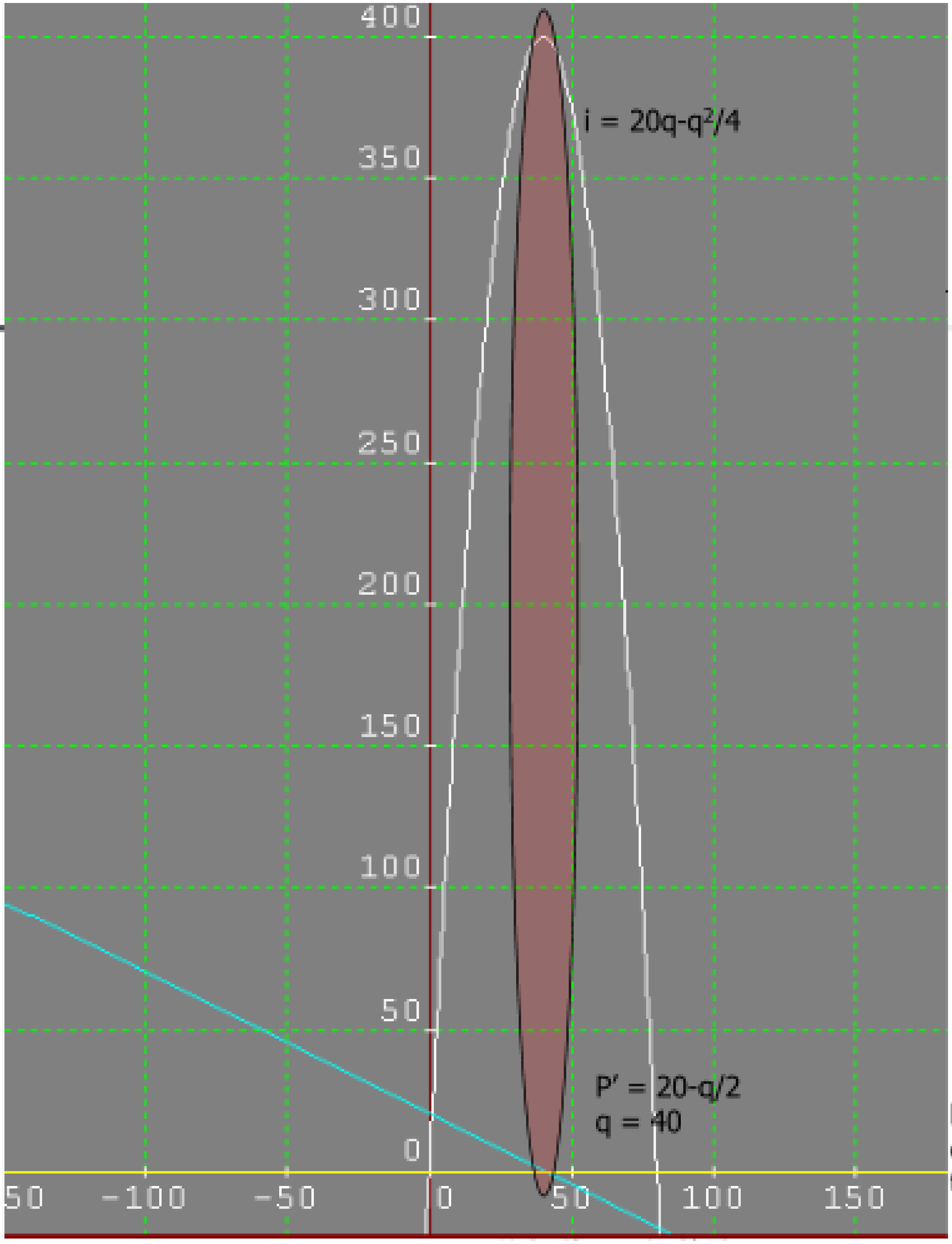
2
y=x⁴/4-x²/2
y'=x³ - x
y''=3x² - 1

Example:

- The price of certain product is given by the following function:

$$p = \frac{80 - q}{4}$$

- Where p is the price y q is the amount sold.
- Fin the value of q that generate a maximum income, considering that income pq.



$P'' = -1/2$
$q \rightarrow$ makes income
to be maximized

WolframAlpha computational... knowledge engine

maximize 20x-x^2/4

Examples Random

Input interpretation:

maximize	$20\,x - \frac{x^2}{4}$

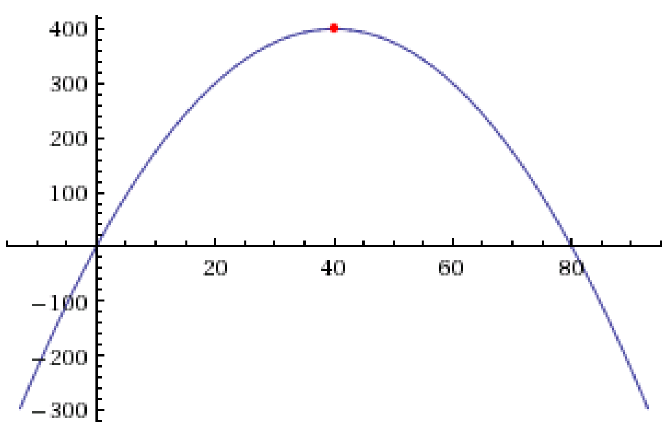
Global maxima:

(no global maxima found)

Local maximum:

$$\max\left\{20\,x - \frac{x^2}{4}\right\} = 400 \text{ at } x = 40$$

Plot:



(x from −12.8 to 92.8)

The case of a bivariate model

- Ifi $z = f(x, y)$ has a relative maxima and/or minima in (x^*, y^*) and if $f'_x(x, y)$ y $f'_y(x, y)$ are defined around (x^*, y^*), then:

- (x^*, y^*) will be a critical point in $f(x, y)$ if they a solution of the system:

$$\begin{pmatrix} f'_x(x, y) = 0 \\ f'_y(x, y) = 0 \end{pmatrix}$$

- Let $D(x, y) = f''_x(x, y)\, f''_y(x,y) - [f'_{xy}(x, y)]^2$
- If:

$$D(x^*, y^*) = \begin{cases} > 0 \text{ y } f''_x(x^*, y^*) < 0,\ f(x, y) \text{ has a max in } x^*, y^* \\ > 0 \text{ y } f''_x(x^*, y^*) > 0,\ f(x, y) \text{ has a min in } x^*, y^* \\ < 0\ f(x, y),\ x^*, y^* \text{ is a saddle point} \\ = 0\ f(x, y) \text{ additional analysis is required} \end{cases}$$

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maximize 5 + 3x - 4y - x^2 + x y - y^2

Examples Random

Input interpretation:

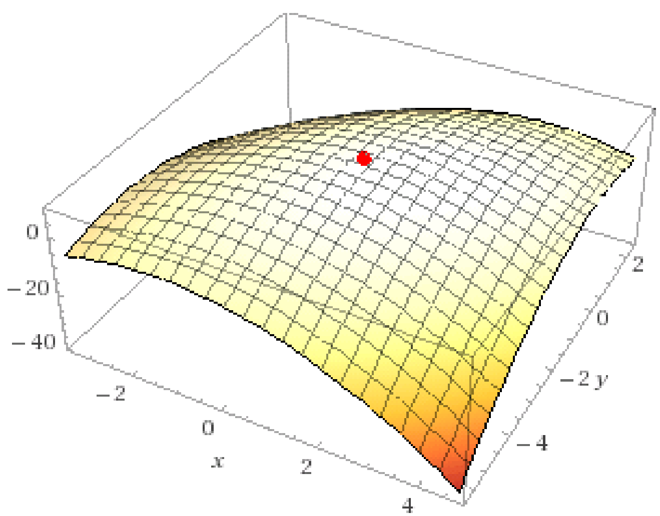
maximize	$5 + 3x - 4y - x^2 + x y - y^2$

Global maximum:

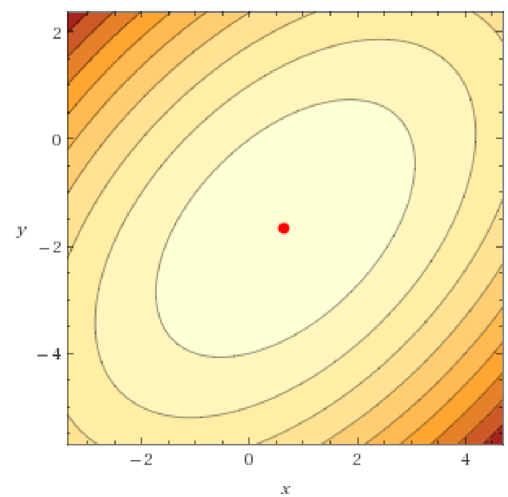
Approximate form

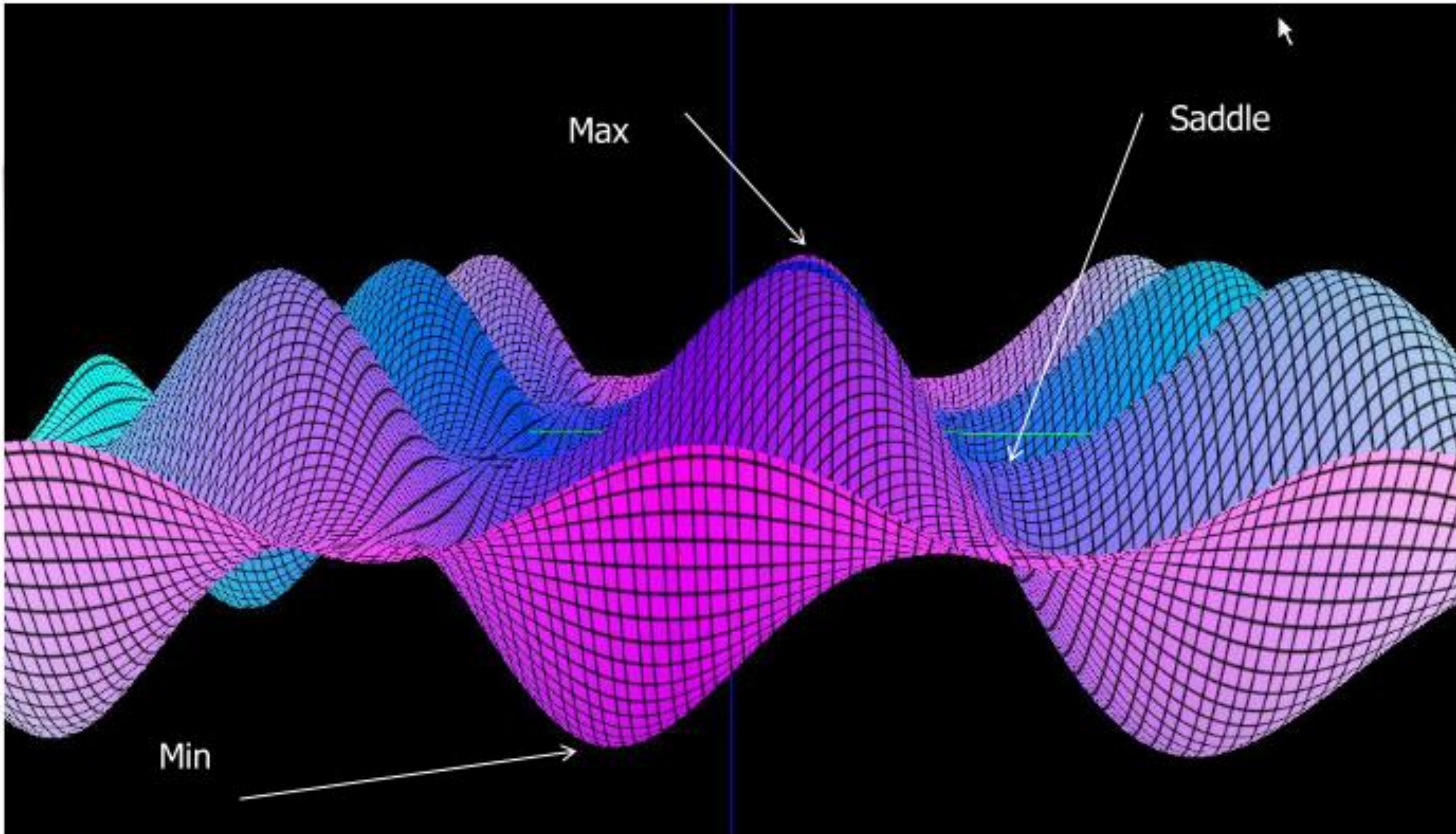
$$\max\{5 + 3x - 4y - x^2 + x y - y^2\} = \frac{28}{3} \text{ at } (x, y) = \left(\frac{2}{3}, -\frac{5}{3}\right)$$

3D plot:

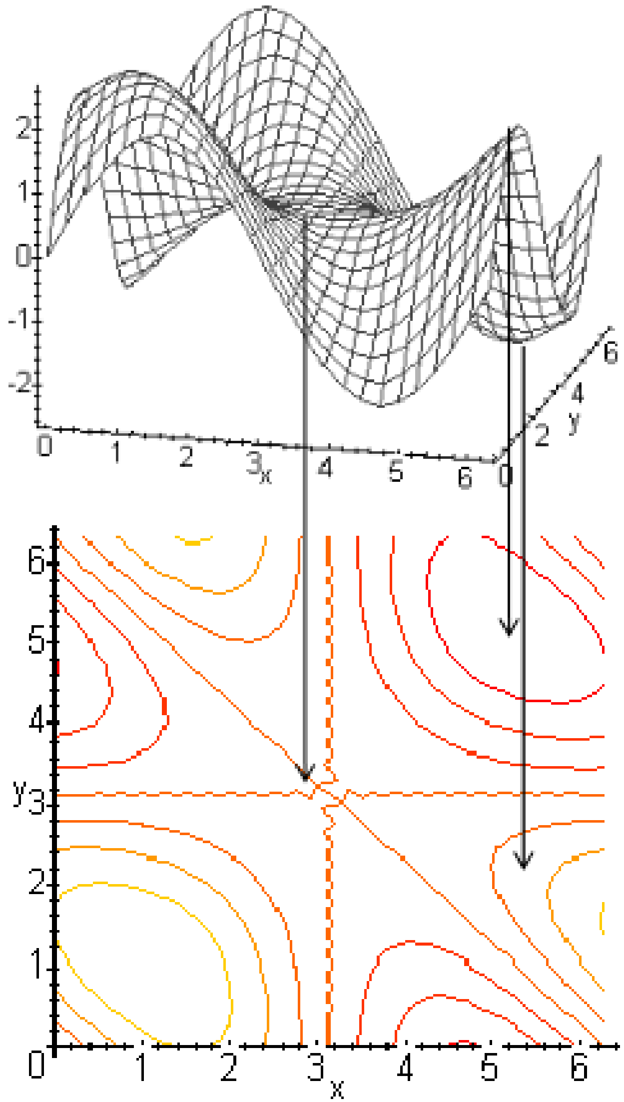


Contour plot:





$$f(x,y) = seno(x) + seno(y) + seno(x+y)$$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The general optimization problem: the unconstrained non linear optimization problem.

- Maximize $f(x)$
- Subject to:
  - $g_i(x) \leq c_i$ i = 1, …, m
- Where f and g_i are general functions of $x \in \Re^n \geq 0$
- If f is convex , and g_i is concave, we have a convex programming problem.

Kuhn-Tucker conditions

- To have an otpimal solution, the problem needs to achieve, as necessary, the Kuhn-Tucker conditions:
- Let $\mathcal{L}$ be the Lagrangian of the maximization function such that:

$$\mathcal{L} = f(\mathbf{x}) + \lambda_1(c_1 - g_1(\mathbf{x})) + \cdots + \lambda_m(c_m - g_m(\mathbf{x}))$$

- Then it has to accomplish the following:

$$\begin{array}{ccc} \dfrac{\partial \mathcal{L}}{\partial x_i} \leq 0 & x_i \geq 0 & x_i \dfrac{\partial \mathcal{L}}{\partial x_i} = 0 \\ g_j(\mathbf{x}) \leq c_j & \lambda_j \geq 0 & \lambda_j(c - g_j(\mathbf{x})) = 0 \end{array}$$

- For every i, j. The variable λ_i is known as the Lagrange coefficient for $\mathcal{L}$

Example

- Optimize

$f(x, y) = xy$

Subject to:

$x^2 + y^2 = 1$

After applying the Lagrangian

$L = xy + \lambda(x^2 + y^2 - 1)$

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d/dx xy + zx^2+zy^2-z

Examples Random

Derivative: Step-by-step solution

$$\frac{\partial}{\partial x}\left(x\,y + z\,x^2 + z\,y^2 - z\right) = 2\,x\,z + y$$

d/dy xy + zx^2+zy^2-z

Examples Random

Derivative: Step-by-step solution

$$\frac{\partial}{\partial y}\left(x\,y + z\,x^2 + z\,y^2 - z\right) = x + 2\,y\,z$$

d/dz xy + zx^2+zy^2-z

Examples Random

Derivative: Step-by-step solution

$$\frac{\partial}{\partial z}\left(x\,y + z\,x^2 + z\,y^2 - z\right) = x^2 + y^2 - 1$$

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solve 2zx+y=0; x+2zy=0; x^2+y^2-1=0

Examples Random

Input interpretation:

solve	$2zx+y=0$
	$x+2zy=0$
	$x^2+y^2-1=0$

Results: More digits

$x=-\frac{1}{\sqrt{2}}\approx-0.707107$ and $y=-\frac{1}{\sqrt{2}}\approx-0.707107$ and $z=-\frac{1}{2}\approx-0.500000$

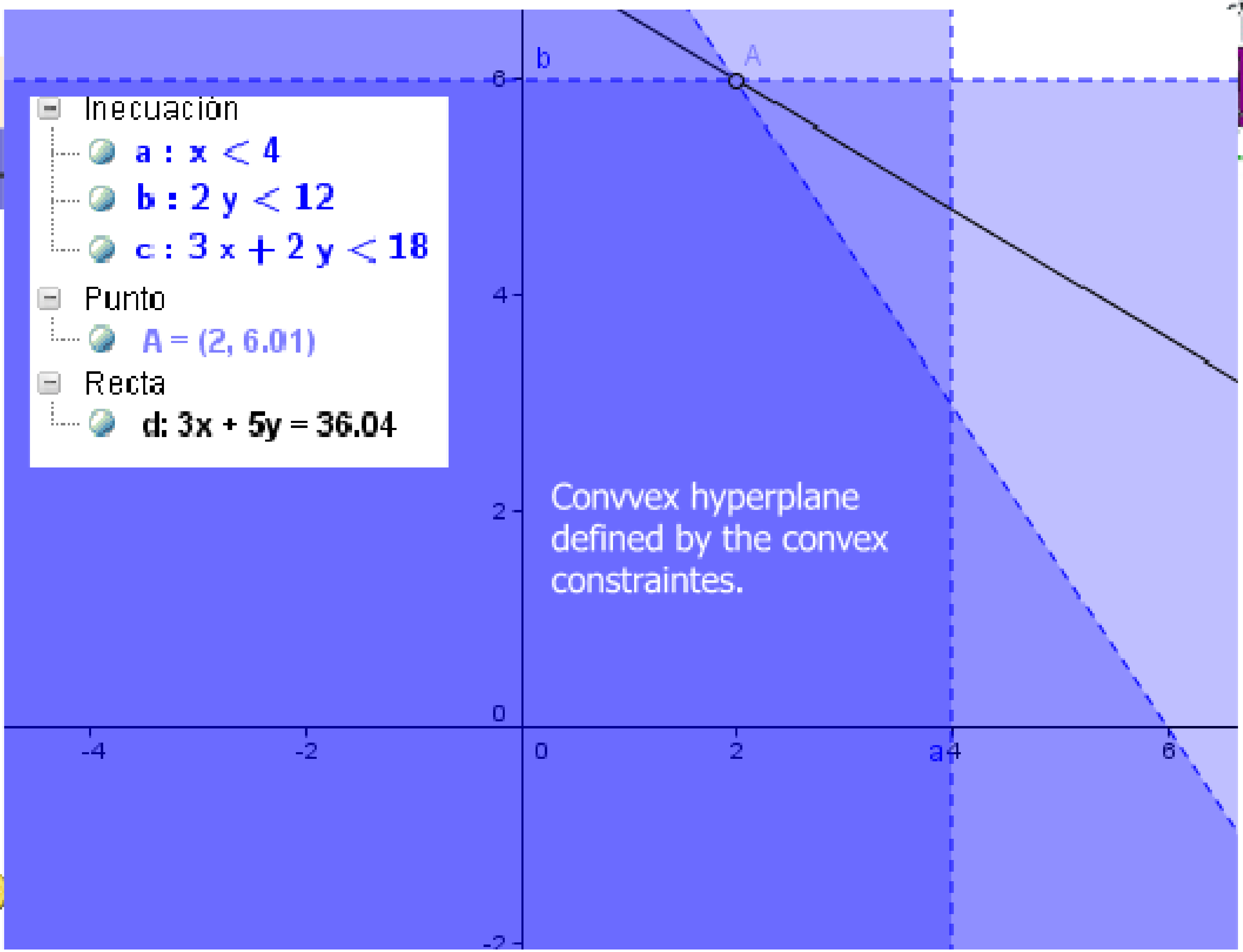
$x=-\frac{1}{\sqrt{2}}\approx-0.707107$ and $y=\frac{1}{\sqrt{2}}\approx0.707107$ and $z=\frac{1}{2}\approx0.500000$

$x=\frac{1}{\sqrt{2}}\approx0.707107$ and $y=-\frac{1}{\sqrt{2}}\approx-0.707107$ and $z=\frac{1}{2}\approx0.500000$

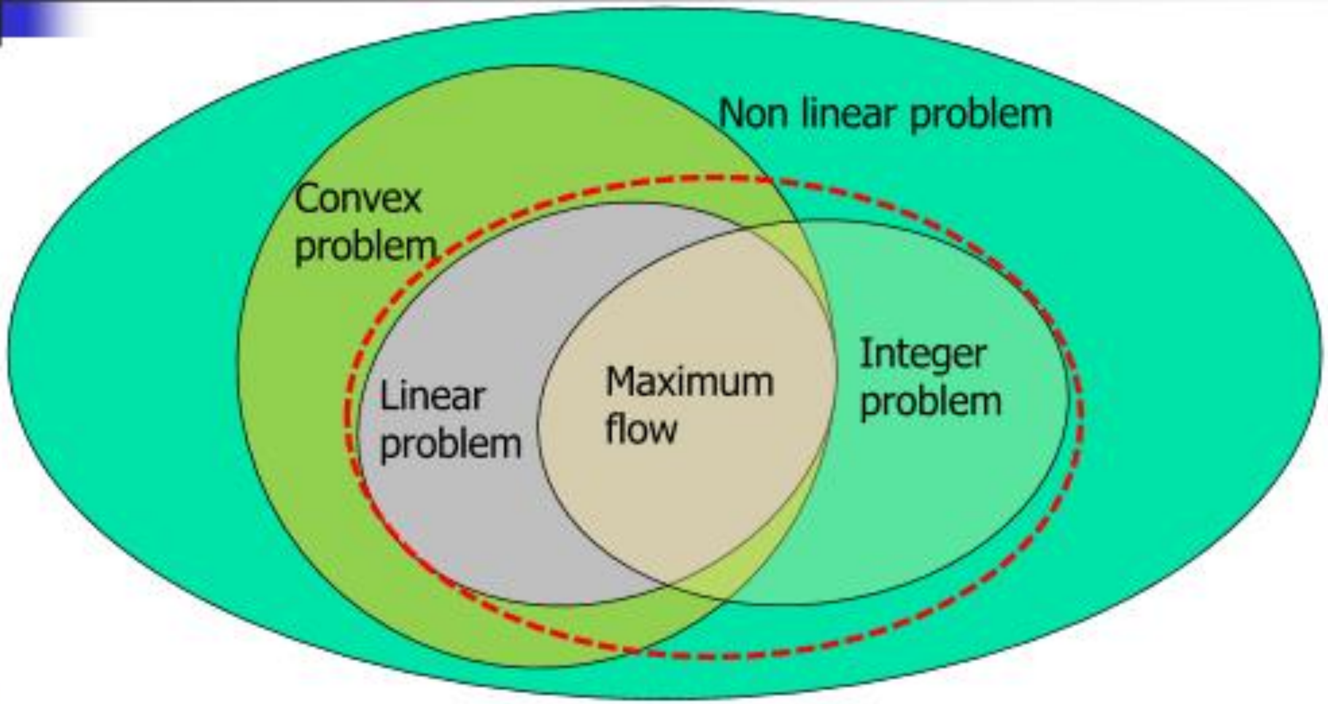
$x=\frac{1}{\sqrt{2}}\approx0.707107$ and $y=\frac{1}{\sqrt{2}}\approx0.707107$ and $z=-\frac{1}{2}\approx-0.500000$

Linear programming

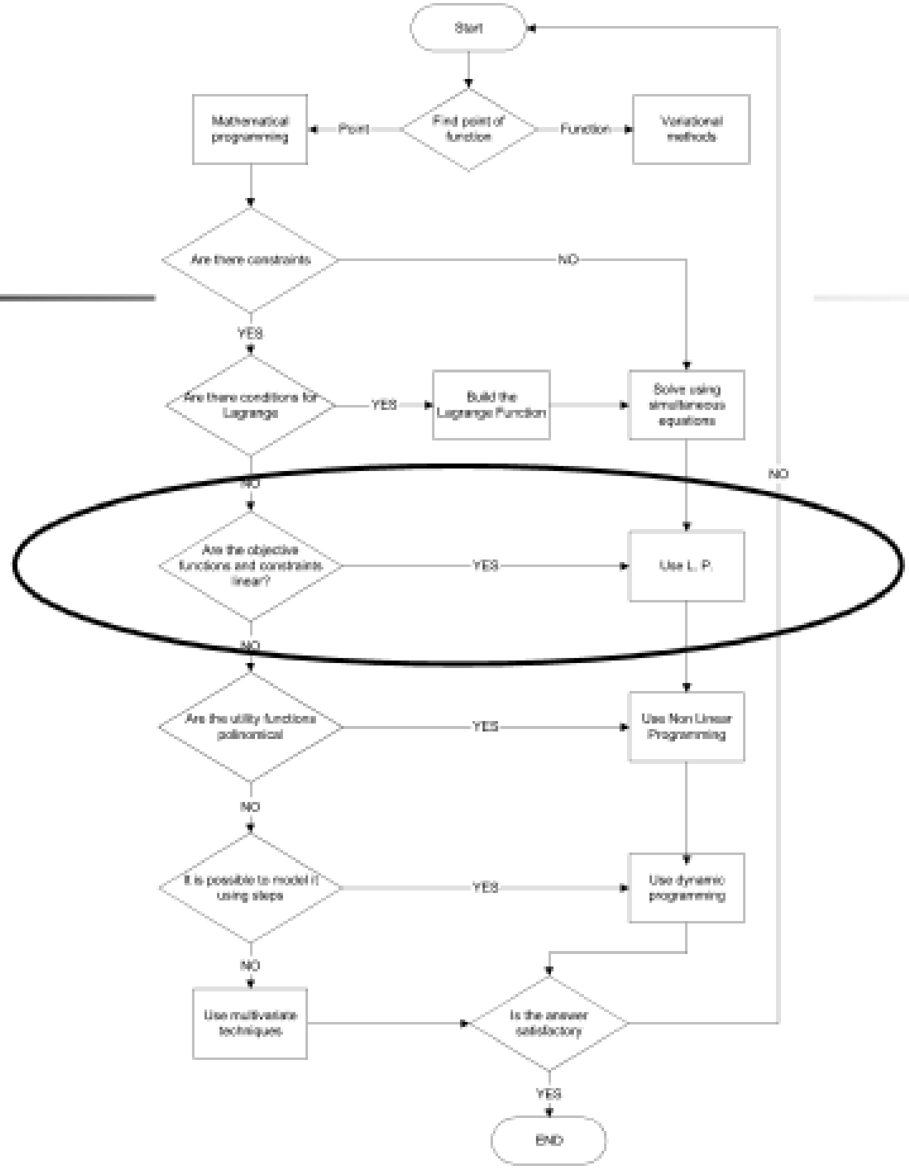
- If f and g_i are lineal and convex, we have a linear programming problem.
- Characteristics:
  - The number of solutions is reduced to a finite number.
  - It is a combinatory problem since all the possible solutions are in the intersections of a convex hyperplane defined by the convex constraints.


Inecuación
a : x < 4
b : 2 y < 12
c : 3 x + 2 y < 18
Punto
A = (2, 6.01)
Recta
d: 3x + 5y = 36.04
Convvex hyperplane
defined by the convex
constraintes.
b
A
a
-4
-2
0
2
4
6
-2

Hierarchy of models



Solutions approaches



How do you solve them?

- Counting all the possible solutions: $\frac{(n-1)!}{2}$
- Finding a "relatively good" solution but with no guarantee that it is the best
- Facing smaller problems being certain that it is possible to find the best solution.
- Using "intelligent search methods" specially in larger problems.

Possible solution methods

- Complete enumeration
- Analytical
- Numerical methods
- Metaheuristics
- Simulation
  - Discrete
  - Continuous
  - Dynamic

Some tools and heuristics

- Simplex
- Karmakar
- Interior Point
- Hungarian Method
- Optimal network
- Taboo Search
- Simulated annealing
- Genetic algorithms
- Greedy algorithms
- Neural networks
- Fuzzy sets
- Ant colony optimization
- Agent based simulation
- Random search

Formulation of an optimization problem

Three basic components are required

- An economic model representing profits or costs – objective function
- A set of constraints that need to be satisfied to solve the model
- An optimization procedure that will locate the values of the independent variables

An example

- Imagine that you have a 5-week business commitment between City A and City B. You fly out of A on Saturdays and return on Mondays. A regular round-trip ticket costs $400, but the price is $ 320 if the dates of the ticket span a weekend. A one-way ticket in either direction costs $ 300. How should you buy the tickets for the 5-week period?
- This is a decision-making problem which requires answering:
  - What are the decision **alternatives**?
  - Under what **restrictions** is the decision made?
  - What is an appropriate **objective criterion** for evaluating the alternatives?

An example

- Consider the alternatives:
  - Buy five regular A-B-A for departure on Saturday and return on Monday of the same week.
  - Buy one A-B, four B A B that span weekends, and one B-A.
  - Buy one A-B-A to cover Saturday of the first week and Monday of the last week and four B-A-B to cover the remaining legs. All tickets in this alternative span at least one weekend.
- The restriction on these options is that you should be able to leave A on Saturday and return on Monday of the same week.

An example

- The objective criterion for evaluating the proposed alternative is the price of the tickets. The alternative that yields the smallest cost is the best:
  - Alternative 1 cost = 5 X 400 = $2000
  - Alternative 2 cost 300 + 4 X 320 + 300 = $1880
  - Alternative 3 cost = 5 X 320 = $1600
- Choose alternative 3.